

Asymptotic teleportation scheme as a universal programmable quantum processor

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We consider a scheme of quantum teleportation where a receiver has multiple (N) output ports and obtains the teleported state by merely selecting one of the N ports according to the outcome of the sender's measurement. We demonstrate that such teleportation is possible by showing an explicit protocol where N pairs of maximally entangled qubits are employed. The optimal measurement performed by a sender is the square-root measurement, and a perfect teleportation fidelity is asymptotically achieved for a large N limit. Such asymptotic teleportation can be utilized as a universal programmable processor.

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Quantum teleportation [1] is a technique to transfer an unknown quantum state from a sender (Alice) to a receiver (Bob) exploiting their prior shared entangled state. In the standard teleportation scheme, Alice first performs a joint measurement on the state to be teleported and half of their entangled state. She then tells the outcome to Bob via a classical communication channel. To complete the teleportation, Bob applies a unitary transformation, which depends on the outcome of the Alice's measurement, to the remaining half of their entangled state.

On the other hand, programmable processors (in short, processors) [2, 3] are devices to manipulate a state via *program states*. Suppose that we wish to apply operation ε to an input state $|\chi_{\text{in}}\rangle$ such that $|\chi_{\text{in}}\rangle \rightarrow \varepsilon(|\chi_{\text{in}}\rangle)$. To do this by using a processor, we first generate the program state $|\varepsilon\rangle$, in which ε is stored. A processor then performs a fixed operation G and accomplishes the desired task such that $G(|\chi_{\text{in}}\rangle \otimes |\varepsilon\rangle) = \varepsilon(|\chi_{\text{in}}\rangle) \otimes |\varepsilon'\rangle$, just like a general-purpose computer executes a program stored in memory. In this way, a programmable processor provides the scheme of storing and retrieving operations. If a processor can deal with arbitrary ε , it is called a universal (programmable) processor. It was shown that a faithful [the output state is exactly $\varepsilon(|\chi_{\text{in}}\rangle)$] and deterministic (with a unit success probability) universal processor cannot be realized by a finite dimensional system [2]. The standard teleportation scheme provides a probabilistic universal processor [2], but the success probability becomes extremely small if the dimension of an input state is large; the obstacle is that Bob's unitary transformation in the teleportation scheme generally does not commute with ε [2, 4].

Let us then consider the teleportation scheme proposed by Knill, Laflamme, and Milburn (KLM) [5] (and its deterministic version [6]), which is a technique to enable linear-optics quantum computation. In the KLM scheme, Bob has multiple (N) output ports and obtains the teleported state by selecting one of the N ports ac-

cording to the outcome of Alice's measurement (see Fig. 1). To complete the teleportation, however, Bob further needs to apply a unitary transformation (phase shift) to the state of the selected port, as well as the standard teleportation scheme. If the KLM scheme is successfully modified such that the unitary transformation is unnecessary (i.e., the state of one of the N ports becomes the teleported state as it is), the teleportation scheme can provide a universal processor. Suppose that Bob applies ε to every port (denoted by $\varepsilon^{\otimes N}$; see Fig. 1) in advance of the teleportation (this corresponds to the operation for storing ε). The teleportation procedure then results in the state processed by ε , regardless of which port is selected. This is because the operation of selecting a port (without any additional unitary transformation) always commutes with $\varepsilon^{\otimes N}$, i.e., selecting a port after applying $\varepsilon^{\otimes N}$ causes the same result as applying $\varepsilon^{\otimes N}$ after selecting a port. This implies that the fixed operation of the teleportation (Alice's measurement and Bob's selection) can *execute* arbitrary ε (including measurements and even trace-nonpreserving operations), if the state $|\psi\rangle$ employed for the teleportation is changed into $|\varepsilon\rangle = (\mathbb{1} \otimes \varepsilon^{\otimes N})|\psi\rangle$. Note that, since the form of the program state $|\varepsilon\rangle$ is known for given ε , we can also generate it by various methods other than applying $\varepsilon^{\otimes N}$.

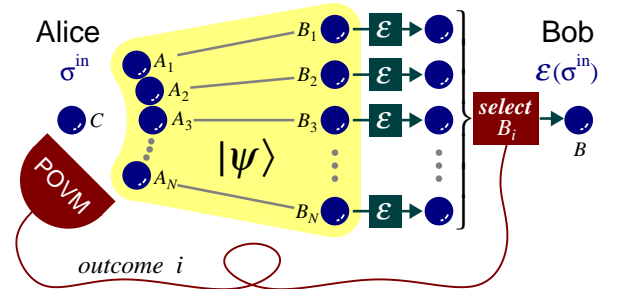


FIG. 1: Setting of asymptotic teleportation.

[7]. This teleportation scheme does not contradict the law of physics that prohibits superluminal (faster than light) communication because without knowing the outcome of Alice's measurement, Bob cannot know which port contains the teleported state and, hence, cannot obtain any information about the teleported state. However, such a scheme must be an approximate one if N is finite; otherwise a faithful and deterministic universal processor would be realized by a finite dimensional system, which contradicts the no-go theorem in [2]. Therefore, it is quite desirable to achieve faithful teleportation in the asymptotic limit of $N \rightarrow \infty$.

In this paper, we demonstrate that such asymptotic teleportation is possible by showing an explicit protocol where N pairs of maximally entangled qubits (quantum bits) are employed. A perfect teleportation fidelity is achieved in the asymptotic limit. Moreover, we determine the optimal measurement performed by Alice.

Now, let us formulate asymptotic teleportation that aims to teleport an unknown state on a qudit (d -dimensional system). To begin with, Alice and Bob share a pure entangled state $|\psi\rangle$ on $2N$ qudits (Fig. 1). Bob has half of the $2N$ qudits: B_1, B_2, \dots , and B_N , where each corresponds to the output port, i.e., the unknown state of Alice's C qudit is finally teleported to one of the N qudits. Alice has the remaining half of the $2N$ qudits: A_1, A_2, \dots , and A_N . These N qudits are denoted by A as a whole. Without loss of generality, $|\psi\rangle$ can be written as

$$|\psi\rangle = (O_A \otimes \mathbb{1}_{B_1 \dots B_N}) |\phi^+\rangle_{A_1 B_1} |\phi^+\rangle_{A_2 B_2} \dots |\phi^+\rangle_{A_N B_N},$$

where $|\phi^+\rangle = (1/\sqrt{d}) \sum_{k=0}^{d-1} |kk\rangle$, and O is an arbitrary operator that satisfies $\text{tr} O O^\dagger = d^N$ so that $|\psi\rangle$ is normalized. Alice then performs a joint measurement with N possible outcomes $(1, 2, \dots, N)$ on the A and C qudits. The measurement is described by a positive operator valued measure (POVM) whose elements are $\{\Pi_i\}$ such that $\sum_{i=1}^N \Pi_i = \mathbb{1}_{AC}$. Suppose that she obtains the outcome i . She then tells the outcome to Bob via a classical channel. Finally, Bob discards the $(N-1)$ qudits of $B_1 B_2 \dots B_{i-1} B_{i+1} \dots B_N$ (i.e., all his qudits except for B_i), which are briefly denoted by \bar{B}_i . The state of the remaining B_i qudit (regarded as B) is the teleported state.

The channel of the above asymptotic teleportation, which maps the density matrices acting on the Hilbert space \mathcal{H}_C to those on \mathcal{H}_B , is thus

$$\begin{aligned} \Lambda(\sigma^{\text{in}}) &= \sum_{i=1}^N \left[\text{tr}_{A\bar{B}_i C} \sqrt{\Pi_i} (|\psi\rangle\langle\psi| \otimes \sigma_C^{\text{in}}) \sqrt{\Pi_i}^\dagger \right]_{B_i \rightarrow B} \\ &= \sum_{i=1}^N \text{tr}_{AC} \Pi_i \left([(O \otimes \mathbb{1}) \sigma_{AB}^{(i)} (O^\dagger \otimes \mathbb{1})] \otimes \sigma_C^{\text{in}} \right) \end{aligned}$$

with

$$\begin{aligned} \sigma_{AB}^{(i)} &= [\text{tr}_{\bar{B}_i} (P_{A_1 B_1}^+ \otimes P_{A_2 B_2}^+ \otimes \dots \otimes P_{A_N B_N}^+)]_{B_i \rightarrow B} \\ &= \frac{1}{d^{N-1}} P_{A_i B}^+ \otimes \mathbb{1}_{\bar{A}_i}, \end{aligned} \quad (1)$$

where $P^+ = |\phi^+\rangle\langle\phi^+|$, and \bar{A}_i is a shorthand notation for $A_1 A_2 \dots A_{i-1} A_{i+1} \dots A_N$. The channel is characterized by the fidelity f averaged over all uniformly distributed input pure states, which is given by $f = (Fd+1)/(d+1)$ with F being the entanglement fidelity of the channel [8]. For the channel Λ , we have

$$\begin{aligned} F &= \text{tr} P_{BD}^+ [(\Lambda \otimes \mathbb{1}) P_{CD}^+] \\ &= \text{tr} \sum_{i=1}^N P_{BD}^+ \Pi_{iAC} \left([(O \otimes \mathbb{1}) \sigma_{AB}^{(i)} (O^\dagger \otimes \mathbb{1})] \otimes P_{CD}^+ \right) \\ &= \frac{1}{d^2} \sum_{i=1}^N \text{tr} \Pi_{iAB} [(O \otimes \mathbb{1}) \sigma_{AB}^{(i)} (O^\dagger \otimes \mathbb{1})]. \end{aligned} \quad (2)$$

Note that Π_i is changed into an operator acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ in the last equality of Eq. (2) because we used the relationship that $(V \otimes \mathbb{1}) P^+ = (\mathbb{1} \otimes V^T) P^+$ for any operator V . Hereafter, the subscript of AB in both Π_i and $\sigma^{(i)}$ is omitted for simplicity, unless it is confusing.

Let us first consider the important case where $O = \mathbb{1}$ and $d = 2$, i.e., N pairs of maximally entangled qubits are employed for asymptotic teleportation. The entanglement fidelity F in this case is $F = (1/4) \sum_{i=1}^N \text{tr} \Pi_i \sigma^{(i)}$. Therefore, the problem of maximizing F with respect to $\{\Pi_i\}$ is equivalent to the quantum detection problem of minimizing the error probability ($p_e = 1 - 4F/N$) of the quantum signals $\{\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(N)}\}$ with equal prior probability $1/N$. The signal states, $\sigma^{(i)}$'s, given by Eq. (1) are mutually non-commutable mixed states, and therefore determining the optimal detection measurement in an analytical way is not easy. Fortunately, however, it can be shown that the square-root measurement (SRM) (also known as a pretty good measurement or least-squares measurement) [9, 10] is indeed optimal for $\{\sigma^{(i)}\}$.

The POVM elements of SRM are given by

$$\Pi_i^{\text{SQ}} = \rho^{-\frac{1}{2}} \sigma^{(i)} \rho^{-\frac{1}{2}} \quad \text{with} \quad \rho = \sum_{i=1}^N \sigma^{(i)}.$$

Since ρ is not full rank, ρ^{-1} is defined on the support of ρ . Moreover, we implicitly assume that $\Delta = (\mathbb{1} - \sum_{i=1}^N \Pi_i^{\text{SQ}})/N$ is added to every Π_i^{SQ} so that the POVM elements sum to identity. Note that the excess term Δ does not affect the entanglement fidelity because $\text{tr} \sigma^{(i)} \Delta = 0$.

Based on the obvious correspondence between qubits and $1/2$ spins, $|0(1)\rangle \leftrightarrow |\frac{1}{2}, -\frac{1}{2}(\frac{1}{2})\rangle$, we regard each qubit as a $1/2$ spin, i.e., $SU(2)$ basis. It is then convenient to consider $|\psi\rangle$ of N pairs of spin singlets, i.e., $|\psi\rangle = |\psi^-\rangle^{\otimes N}$ (instead of $|\psi\rangle = |\phi^+\rangle^{\otimes N}$), and as a result P^+ in $\sigma^{(i)}$ is replaced by $P^- = |\psi^-\rangle\langle\psi^-|$ where

$|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. The POVM elements for the two cases are easily interconverted by applying the unitary transformation ($\mathbb{1}_A \otimes \sigma_y$). In the language of $SU(2)$ representation, eigenvectors with eigenvalues $\lambda_j^- = (N/2 - j)/2^N$ and $\lambda_j^+ = (N/2 + j + 1)/2^N$ of ρ are given by

$$|\Psi_{\mp}^{[N]}(\lambda_j^{\mp}; m, \alpha)\rangle = \pm |0\rangle_B |\Phi^{[N]}(j, m_+, \alpha)\rangle |j, m_+; j_{\pm}\rangle_- \\ \pm |1\rangle_B |\Phi^{[N]}(j, m_-, \alpha)\rangle |j, m_-; j_{\pm}\rangle_+,$$

where $|\Phi^{[N]}(j, m, \alpha)\rangle = |j, m, \alpha\rangle$ denotes the orthogonal basis of N -spin systems, i.e., the basis of irreducible representation of $SU(2)^{\otimes N}$. The spin angular momentum j runs from j_{\min} to $N/2$ ($m = -j, \dots, j$), where $j_{\min} = 0$ ($1/2$) when N is even (odd), and α specifies the additional degree of freedom. Here, we introduced a shorthand notation for (nonvanishing) Clebsch-Gordan coefficients,

$$\langle j_1, m_1; j \rangle_{\pm} = \langle j_1, m_1, \frac{1}{2}, \pm \frac{1}{2} | j, m_1 \pm \frac{1}{2} \rangle \\ = (-1)^{j_1 + \frac{1}{2} - j} \langle \frac{1}{2}, \pm \frac{1}{2}, j_1, m_1 | j, m_1 \pm \frac{1}{2} \rangle,$$

and write $j_{\pm} = j \pm \frac{1}{2}$ and $m_{\pm} = m \pm \frac{1}{2}$. The proof of the eigenvalue equation

$$\rho |\Psi_{\mp}^{[N]}(\lambda_j^{\mp}; m)\rangle = \lambda_j^{\mp} |\Psi_{\mp}^{[N]}(\lambda_j^{\mp}; m)\rangle. \quad (3)$$

is carried out by induction and by noting that $\rho = \rho^{[N]}$ is constructed recursively:

$$\rho^{[N]} = \rho^{[N-1]} \otimes \frac{\mathbb{1}_{A_N}}{2} + \frac{\mathbb{1}_{A_1}}{2} \otimes \dots \otimes \frac{\mathbb{1}_{A_{N-1}}}{2} \otimes P_{BA_N}^-.$$

Details are presented in Appendix A.

The N -spin eigenfunctions $|\Phi^{[N]}\rangle$ are computed recursively: $|\Phi^{[N-1]}\rangle |\Phi^{[1]}\rangle \rightarrow |\Phi^{[N]}\rangle$, where $|\Phi^{[N-1]}\rangle$ are $(N-1)$ -spin eigenfunctions of the first $(N-1)$ spins (A_1, \dots, A_{N-1}) and $|\Phi^{[1]}\rangle$ are the $1/2$ spin function of the A_N qubit. The other choice of construction of $|\Phi^{[N]}\rangle$ results in a different set of functions, $|\Phi^{[N]'}(j, m, \alpha')\rangle$, which are unitarily equivalent to $|\Phi^{[N]}(j, m, \alpha)\rangle$, and the unitary transformation depends only on α and α' for each j . This fact enables us to calculate explicitly the matrix elements involved in the calculation of the entanglement fidelity F as follows:

$$\langle \xi^{(i)}(s, s_z, \alpha) | \rho^{-\frac{1}{2}} | \xi^{(i)}(s', s'_z, \alpha') \rangle = \delta_{s, s'} \delta_{s_z, s'_z} \delta_{\alpha, \alpha'} c(s) \quad (4)$$

with

$$c(s) = \left(\lambda_{s-\frac{1}{2}}^- \right)^{-\frac{1}{2}} \frac{s}{2s+1} + \left(\lambda_{s+\frac{1}{2}}^+ \right)^{-\frac{1}{2}} \frac{s+1}{2s+1}. \quad (5)$$

Here, the states of

$$|\xi^{(i)}(s, s_z, \alpha)\rangle = |\psi^-\rangle_{BA_i} |\Phi^{[N-1]'}(s, s_z, \alpha)\rangle$$

with $|\Phi^{[N-1]'}\rangle$ being the $(N-1)$ -spin eigenfunction for the \bar{A}_i qubits, are the eigenfunctions of $\sigma^{(i)}$, and thus,

$$\sigma^{(i)} = \frac{1}{2^{N-1}} \sum_{s=s_{\min}}^{(N-1)/2} \sum_{s_z, \alpha} |\xi^{(i)}(s, s_z, \alpha)\rangle \langle \xi^{(i)}(s, s_z, \alpha)|,$$

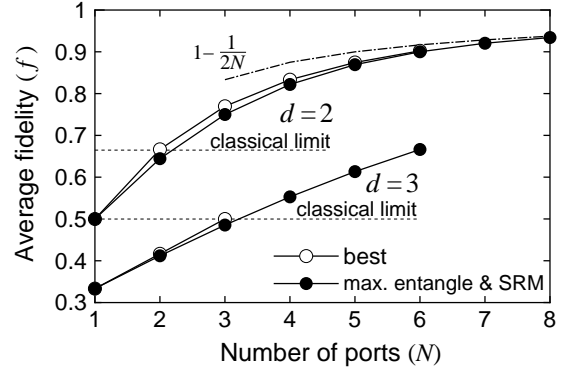


FIG. 2: Average fidelity (f) of asymptotic teleportation as function of number of output ports (N).

where $s_{\min} = 0$ ($1/2$) when $N-1$ is even (odd). Note that matrix elements of Eq. (4) depend only on s . See Appendix B for the detailed derivations of Eqs. (4) and (5). Note further that, in terms of $|\xi^{(i)}(s, s_z, \alpha)\rangle$, both ρ and $\sigma^{(i)}$ are found to be block-diagonal with respect to s . The block matrices are denoted by $\rho(s)$ and $\sigma^{(i)}(s)$, respectively.

Now, from Eqs. (4) and (5), we have

$$F = \frac{1}{2^2} \sum_{s=s_{\min}}^{(N-1)/2} \sum_{i=1}^N \text{tr} \rho(s)^{-\frac{1}{2}} \sigma^{(i)}(s) \rho(s)^{-\frac{1}{2}} \sigma^{(i)}(s) \\ = \frac{N}{2^{2N}} \sum_{s=s_{\min}}^{(N-1)/2} \sum_{s_z, \alpha} c(s)^2 \\ = \frac{N}{2^{2N}} \sum_{s=s_{\min}}^{(N-1)/2} \frac{(2s+1)^2 (N-1)!}{\left(\frac{N-1}{2} - s\right)! \left(\frac{N+1}{2} + s\right)!} c(s)^2 \\ = \frac{1}{2^{N+3}} \sum_{k=0}^N \left(\frac{N-2k-1}{\sqrt{k+1}} + \frac{N-2k+1}{\sqrt{N-k+1}} \right)^2 \binom{N}{k}.$$

Here, we introduced $k = (N-1)/2 - s$ in the last equality. The corresponding average fidelity f as a function of N is plotted by closed circles in Fig. 2. For $N \geq 3$, the fidelity exceeds the classical limit $f_{\text{cl}} = 2/(d+1)$ ($f_{\text{cl}} = 2/3$ for $d=2$), which is the best fidelity via a classical channel only [8]. Therefore, this protocol works as quantum teleportation for $N \geq 3$. Moreover, the fidelity approaches to $f = 1$ for increasing N . In fact, by expanding $(1-x)^{-1/2}$ in F into the Taylor series of $\frac{(N-2k)^2}{(N+2)^2}$ and noting $y(m) \equiv \sum_{k=0}^N (N-2k)^{2m} \binom{N}{k} = \mathcal{O}(N^m) 2^N = [(x \frac{d}{dx})^{2m} (x + \frac{1}{x})^N]_{x=1}$ [with $y(1) = N 2^N$ and $y(2) = N(3N-2) 2^N$], we find that $f \rightarrow 1 - 1/(2N)$ for $N \rightarrow \infty$. Therefore, the protocol of employing N spin singlets and SRM certainly achieves perfect fidelity in the asymptotic limit.

Let us then prove that SRM is an optimal measurement for $|\psi\rangle$ of N spin singlets. The problem of maximizing $F = (1/4) \sum_{i=1}^N \text{tr} \Pi_i \sigma^{(i)}$ is a semidefinite program [11] and thus has the dual problem of minimizing $(1/4) \text{tr} Y$

subject to $Y - \sigma^{(i)} \geq 0$ for all i [10, 12]. Any feasible solution of the dual problem gives an upper bound of the original problem. Therefore, it is enough to show that $Y^{\text{SQ}} = \sum_{i=1}^N \Pi_i^{\text{SQ}} \sigma^{(i)}$ is a feasible solution (i.e., $Y^{\text{SQ}} - \sigma^{(i)} \geq 0$) because $(1/4)\text{tr}Y^{\text{SQ}}$ agrees with F obtained by SRM. Using Eq. (4), we find that $Y^{\text{SQ}} = \sum_{s=s_{\min}}^{(N-1)/2} Y^{\text{SQ}}(s)$ with $Y^{\text{SQ}}(s) = \frac{c(s)}{2^{N-1}} \rho(s)^{\frac{1}{2}}$. It has been shown that $A - (1/c)|\xi\rangle\langle\xi| \geq 0$ if $|\xi\rangle \in \text{range}(A)$ and $c = \langle\xi|A^{-1}|\xi\rangle$ [13]. Moreover, $A - (1/c)(|\xi\rangle\langle\xi| + |\xi_{\perp}\rangle\langle\xi_{\perp}|) \geq 0$ for $|\xi_{\perp}\rangle$ such that $\langle\xi_{\perp}|\xi\rangle = 0$, $|\xi_{\perp}\rangle \in \text{range}(A)$, and $c = \langle\xi_{\perp}|A^{-1}|\xi_{\perp}\rangle$ because $\langle\xi_{\perp}|[A - (1/c)|\xi\rangle\langle\xi|]^{-1}|\xi_{\perp}\rangle = c$ [13]. Repeating this, it is found that $A - (1/c)\sum_k |\xi_k\rangle\langle\xi_k| \geq 0$ where $|\xi_k\rangle$ are mutually orthogonal vectors such that $|\xi_k\rangle \in \text{range}(A)$ and $\langle\xi_k|A^{-1}|\xi_k\rangle = c$. Therefore,

$$\rho(s)^{\frac{1}{2}} - \frac{1}{c(s)} \sum_{s_z, \alpha} |\xi^{(i)}(s, s_z, \alpha)\rangle\langle\xi^{(i)}(s, s_z, \alpha)| \geq 0$$

follows from Eq. (4), and thus $Y^{\text{SQ}}(s) - \sigma^{(i)}(s) \geq 0$, which completes the proof of the optimality.

Let us return to Eq. (2) and investigate the cases of general O . We need to optimize both $\{\Pi_i\}$ and O to obtain the best fidelity of asymptotic teleportation. By introducing $\tilde{\Pi}_i = (O^{\dagger} \otimes \mathbb{1})\Pi_i(O \otimes \mathbb{1})$ and $X = O^{\dagger}O$, however, the best entanglement fidelity is obtained by maximizing

$$F = \frac{1}{d^2} \sum_{i=1}^N \text{tr} \tilde{\Pi}_{iAB} \sigma_{AB}^{(i)} \quad (6)$$

under the constraints of $\tilde{\Pi}_i \geq 0$, $\sum_{i=1}^N \tilde{\Pi}_i = (X \otimes \mathbb{1})$, $X \geq 0$, and $\text{tr}X = d^N$. This is also a semidefinite program. The dual problem is of minimizing $F = d^{N-2}a$ subject to $\Omega - \sigma^{(i)} \geq 0$ and $a\mathbb{1} - \text{tr}_B \Omega \geq 0$. The constraints are satisfied if we take $\Omega = \sum_{i=1}^N \sigma^{(i)}$ and $a = N/d^N$, and hence we have $F \leq N/d^2$. This upper bound is tight for $N \leq d$. In fact, letting $\{|e_k\rangle\}$ be the set of N orthogonal states on \mathbb{C}^d , the protocol of employing the separable $|\psi\rangle = \bigotimes_{k=1}^N |0\rangle_{A_k} |e_k\rangle_{B_k}$, which results in mutually orthogonal $(O \otimes \mathbb{1})\sigma^{(i)}(O^{\dagger} \otimes \mathbb{1}) = (|0\rangle\langle 0|)_A \otimes (|e_i\rangle\langle e_i|)_B$, achieves the upper bound. The corresponding bound for the average fidelity is $f \leq (d+N)/[d(d+1)] \leq$

f_{cl} , and therefore it is concluded that $N > d$ is necessary for any protocol to exceed the classical limit of fidelity.

For $N > d$, such a construction of N orthogonal states becomes impossible (even using entangled states), and the best F may deviate from N/d^2 . We have solved the semidefinite program Eq. (6) using the numerical package of SDPA [14]. The results for $d = 2$ and $N \leq 6$ are plotted by open circles in Fig. 2. It is found from the figure that the best fidelity is nearly achieved by the protocol of employing spin singlets (maximally entangled $|\psi\rangle$) and SRM. Interestingly, although the difference is small, this implies that a non-maximally entangled $|\psi\rangle$ provides a higher fidelity than that of the maximally entangled $|\psi\rangle$. In Fig. 2, the fidelity for the case of maximally entangled qutrits ($d = 3$) and SRM is also plotted, which was obtained by the numerical diagonalization of ρ . The numerical investigations suggest that SRM is optimal even in this case. For the case of maximally entangled qudits with general d and SRM, it can be shown by the same technique as used in [15] that $f \geq 1 - d(d-1)/N$. Therefore, the protocol provides a perfect fidelity in the asymptotic limit for any d .

To summarize, we considered a scheme of asymptotic quantum teleportation where Bob has multiple output ports and obtains the teleported state by simply selecting one of the N ports. We showed that, if N pairs of maximally entangled qubits are employed, the square-root measurement is the optimal measurement performed by Alice. This protocol provides a perfect fidelity in the asymptotic limit and nearly achieves the best fidelity of asymptotic teleportation. The scheme of asymptotic teleportation provides a universal programmable processor in a simple and natural way: to process a state by operation ε , teleport the state by employing $|\varepsilon\rangle = (\mathbb{1} \otimes \varepsilon^{\otimes N})|\psi\rangle$ instead of $|\psi\rangle$. The fidelity of the processor dealing with trace-preserving ε is always equal to or higher than the fidelity of asymptotic teleportation alone because of the monotonicity such that $f(\varepsilon(|\chi_{\text{in}}\rangle), \varepsilon(\Lambda(|\chi_{\text{in}}\rangle))) \geq f(|\chi_{\text{in}}\rangle, \Lambda(|\chi_{\text{in}}\rangle))$, where Λ is the teleportation channel, and therefore an asymptotically faithful programmable processor is realized.

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APPENDIX A: PROOF OF EQ. (3)

In this appendix, we drop α from $|\Phi^{[N]}([N-1])\rangle$ and $|\Psi^{[N]}([N-1])\rangle$ for simplicity. We note that $|\Phi^{[N]}(j, \dots)\rangle$ is classified into two; one is the linear combination of $|\Phi^{[N-1]}(j_+, \dots)\rangle |i\rangle_{A_N}$ and the other, $|\Phi^{[N-1]}(j_-, \dots)\rangle |i\rangle_{A_N}$. We call the former (latter) is of the type-I (II);

$$|\Phi_{I(II)}^{[N]}(j, m)\rangle = |\Phi^{[N-1]}(j_+(j_-), m_+)\rangle |0\rangle_{A_N} \langle j_+(j_-), m_+; j\rangle_- + |\Phi^{[N-1]}(j_+(j_-), m_-)\rangle |1\rangle_{A_N} \langle j_+(j_-), m_-; j\rangle_+.$$

According to the different types of $|\Phi^{[N]}\rangle$, $|\Psi_{\mp}^{[N]}\rangle$ have two types;

$$\begin{aligned} |\Psi_{\mp I(II)}^{[N]}(\lambda_j^{\mp}, m)\rangle = & \pm |0\rangle_B |\Phi^{[N-1]}(j_+(j_-), m_{++})\rangle |0\rangle_{A_N} \langle j, m_+; j_{\pm}\rangle_- \langle j_+(j_-), m_{++}; j\rangle_- \\ & \pm |0\rangle_B |\Phi^{[N-1]}(j_+(j_-), m)\rangle |1\rangle_{A_N} \langle j, m_+; j_{\pm}\rangle_- \langle j_+(j_-), m; j\rangle_+ \\ & \pm |1\rangle_B |\Phi^{[N-1]}(j_+(j_-), m)\rangle |1\rangle_{A_N} \langle j, m_-; j_{\pm}\rangle_+ \langle j_+(j_-), m; j\rangle_- \\ & \pm |1\rangle_B |\Phi^{[N-1]}(j_+(j_-), m_{--})\rangle |1\rangle_{A_N} \langle j, m_-; j_{\pm}\rangle_+ \langle j_+(j_-), m_{--}; j\rangle_+, \end{aligned} \quad (A1)$$

where $j_{\pm\pm} = j \pm 1$ and $m_{\pm\pm} = m \pm 1$. Since Eq. (3) is obvious for $N = 1$, our aim is to prove Eq. (3) under the assumption that Eq. (3) with $N \rightarrow N - 1$ holds true. To this end, we write $|\Psi^{[N]}\rangle$ in terms of $|\Psi^{[N-1]}\rangle$ as follows.

$$\begin{aligned} |\Psi_{\mp I(II)}^{[N]}(\lambda_j^{\mp}; m)\rangle = & \pm |\Psi_{-}^{[N-1]}(\lambda_{j_+(j_-)}^{\mp}; m_+)\rangle |0\rangle_{A_N} \\ & \times [\langle j_+(j_-), m_{++}; j_{++}(j)\rangle_-^* \langle j, m_+; j_{\pm}\rangle_- \langle j_+(j_-), m_{++}; j\rangle_- + \langle j_+(j_-), m; j_{++}(j)\rangle_+^* \langle j, m_-; j_{\pm}\rangle_+ \langle j_+(j_-), m; j\rangle_-] \\ & \pm |\Psi_{-}^{[N-1]}(\lambda_{j_+(j_-)}^{\mp}; m_-)\rangle |1\rangle_{A_N} \\ & \times [\langle j_+(j_-), m; j_{++}(j)\rangle_-^* \langle j, m_+; j_{\pm}\rangle_- \langle j_+(j_-), m; j\rangle_+ + \langle j_+(j_-), m_{--}; j_{++}(j)\rangle_+^* \langle j, m_-; j_{\pm}\rangle_+ \langle j_+(j_-), m_{--}; j\rangle_+] \\ & \mp |\Psi_{+}^{[N-1]}(\lambda_{j_+(j_-)}^{\mp}; m_+)\rangle |0\rangle_{A_N} \\ & \times [\langle j_+(j_-), m_{++}; j(j--)\rangle_-^* \langle j, m_+; j_{\pm}\rangle_- \langle j_+(j_-), m_{++}; j\rangle_- + \langle j_+(j_-), m; j(j--)\rangle_+^* \langle j, m_-; j_{\pm}\rangle_+ \langle j_+(j_-), m; j\rangle_-] \\ & \mp |\Psi_{+}^{[N-1]}(\lambda_{j_+(j_-)}^{\mp}; m_-)\rangle |1\rangle_{A_N} \\ & \times [\langle j_+(j_-), m; j(j--)\rangle_-^* \langle j, m_+; j_{\pm}\rangle_- \langle j_+(j_-), m; j\rangle_+ + \langle j_+(j_-), m_{--}; j(j--)\rangle_+^* \langle j, m_-; j_{\pm}\rangle_+ \langle j_+(j_-), m_{--}; j\rangle_+]. \end{aligned} \quad (A2)$$

Equation (A2) is obtained by computing the overlap between $|\Psi_{\mp I(II)}^{[N]}\rangle$ given by Eq. (A1) and $|\Psi_{\mp}^{[N-1]}\rangle$ given by Eq. (??) with $N \rightarrow N - 1$. The vector $\rho^{[N-1]} \otimes I_{A_N} |\Psi_{\mp I(II)}^{[N]}(\lambda_j^{\mp}; m)\rangle$ takes the form of the right hand side (r.h.s.) of Eq. (A2) with

$$|\Psi_{\mp}^{[N-1]}(\lambda_{j_+(j_-)}^{\mp}; \dots)\rangle \rightarrow \lambda_{j_+(j_-)}^{[N-1]\mp} |\Psi_{\mp}^{[N-1]}(\lambda_{j_+(j_-)}^{[N-1]\mp}; \dots)\rangle, \quad (A3)$$

while the vector $I_{A_1} \otimes \cdots \otimes I_{A_{N-1}} \otimes P_{BA_N}^- \left| \Psi_{\mp I(II)}^{[N]}(\lambda_j^\mp; m) \right\rangle$ takes the form of the r.h.s. of Eq. (A1) with

$$\begin{aligned} |0\rangle_B |1\rangle_{A_N} &\rightarrow (|0\rangle_B |1\rangle_{A_N} - |1\rangle_B |0\rangle_{A_N}) / \sqrt{2} \\ |1\rangle_B |0\rangle_{A_N} &\rightarrow -(|0\rangle_B |1\rangle_{A_N} - |1\rangle_B |0\rangle_{A_N}) / \sqrt{2}. \end{aligned}$$

In Eq. (A3), we attached a superscript $[N-1]$ to eigenvalues λ^\mp emphasizing the relevant system size. Putting these two results together (and after lengthy calculations), we can see the desired eigenvalue equation,

$$\rho^{[N]} \left| \Psi_{\mp I(II)}^{[N]}(\lambda_j^\mp; m) \right\rangle = \lambda_j^\mp \left| \Psi_{\mp I(II)}^{[N]}(\lambda_j^\mp; m) \right\rangle.$$

This completes the proof.

APPENDIX B: EQ. (4) WITH EQ. (5)

To calculate the entanglement fidelity F , we decompose $\rho^{[N]}$ into operators acting on eigenspace with eigenvalues λ_j^\mp ,

$$\rho_{\mp}^{[N]}(\lambda_j^\mp) = \lambda_j^\mp \sum_{m=-(j\pm 1/2)}^{j\pm 1/2} \sum_{\alpha} \left| \Psi_{\mp}^{[N]}(\lambda_j^\mp, m, \alpha) \right\rangle \left\langle \Psi_{\mp}^{[N]}(\lambda_j^\mp, m, \alpha) \right|, \quad (\text{B1})$$

so that we can write $\rho^{[N]} = \rho_-^{[N]} \oplus \rho_+^{[N]}$, where $\rho_{\mp}^{[N]} = \oplus_j \rho_{\mp}^{[N]}(\lambda_j^\mp)$. Let us recall that $|\Phi^{[N-1]}(s, s_z, \alpha)\rangle$ is unitarily equivalent to $|\Phi^{[N-1]'}(s, s_z, \alpha')\rangle$ and the unitary transform is independent of s_z ;

$$|\Phi^{[N-1]}(s, s_z, \alpha)\rangle = \sum_{\alpha\alpha'} U_{\alpha\alpha'}(s) |\Phi^{[N-1]'}(s, s_z, \alpha')\rangle$$

to obtain

$$\begin{aligned} &\left\langle \xi^{(l)}(s, s_z, \beta) \left| \Psi_{\mp I(II)}^{[N]}(\lambda_j^\mp, m, \alpha) \right\rangle \right. \\ &= \pm \frac{1}{\sqrt{2}} U_{\alpha\beta}(j) \delta_{s, j+(j-)} \delta_{s_z, m} (\langle j, m_+; j_{\pm} \rangle_- \langle j_+(j-), m; j \rangle_+ - \langle j, m_-; j_{\pm} \rangle_+ \langle j_+(j-), m; j \rangle_-). \end{aligned} \quad (\text{B2})$$

From Eqs. (B1) and (B2) we have

$$\begin{aligned} &\left\langle \xi^{(l)}(s, s_z, \beta) \left| \left(\rho_-^{[N]} \right)^{-\frac{1}{2}} \left| \xi^{(l)}(s', s'_z, \beta') \right\rangle \right. \right. \\ &= \frac{1}{2} \delta_{s, s'} \delta_{s_z, s'_z} \delta_{\beta, \beta'} \left(\lambda_{s-}^- \right)^{-\frac{1}{2}} \left| \langle s_-, s_{z+}; s \rangle_- \langle s, s_z; s_- \rangle_+ - \langle s_-, s_{z-}; s \rangle_+ \langle s, s_z; s_- \rangle_- \right|^2 \\ &+ \frac{1}{2} \delta_{s, s'} \delta_{s_z, s'_z} \delta_{\beta, \beta'} \left(\lambda_{s+}^- \right)^{-\frac{1}{2}} \left| \langle s_+, s_{z+}; s \rangle_- \langle s, s_z; s_+ \rangle_+ - \langle s_+, s_{z-}; s \rangle_+ \langle s, s_z; s_+ \rangle_- \right|^2 \\ &= \delta_{s, s'} \delta_{s_z, s'_z} \delta_{\beta, \beta'} \left(\lambda_{s-}^- \right)^{-\frac{1}{2}} \frac{s}{2s+1} \end{aligned} \quad (\text{B3})$$

and

$$\begin{aligned} &\left\langle \xi^{(l)}(s, s_z, \beta) \left| \left(\rho_+^{[N]} \right)^{-\frac{1}{2}} \left| \xi^{(l)}(s', s'_z, \beta') \right\rangle \right. \right. \\ &= \frac{1}{2} \delta_{s, s'} \delta_{s_z, s'_z} \delta_{\beta, \beta'} \left(\lambda_{s-}^+ \right)^{-\frac{1}{2}} \left| \langle s_-, s_{z+}; s \rangle_- \langle s, s_z; s_- \rangle_+ - \langle s_-, s_{z-}; s \rangle_- \langle s, s_z; s_- \rangle_- \right|^2 \\ &+ \frac{1}{2} \delta_{s, s'} \delta_{s_z, s'_z} \delta_{\beta, \beta'} \left(\lambda_{s+}^+ \right)^{-\frac{1}{2}} \left| \langle s_+, s_{z+}; s \rangle_- \langle s, s_z; s_+ \rangle_+ - \langle s_+, s_{z-}; s \rangle_+ \langle s, s_z; s_+ \rangle_- \right|^2 \\ &= \delta_{s, s'} \delta_{s_z, s'_z} \delta_{\beta, \beta'} \left(\lambda_{s+1/2}^+ \right)^{-\frac{1}{2}} \frac{s+1}{2s+1} \end{aligned} \quad (\text{B4})$$

The sum of these two [Eqs. (B3) and (B4)] yields Eq. (4) with Eq. (5).